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The Wiener Polynomial of The Tensor Product

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ABSTRACT

Let $G1$ and $G2$ be vertex disjoint connected graphs such that each edge of $G1$ and $G2$ is a triangle edge. In this paper, the coefficients of the Wiener polynomial of the tensor product $G1\otimes G2$ are determined in terms of the coefficients of $W(G1;x)$ and $W(G2;x)$. The Wiener polynomial of the tensor product of a path graph and an odd cycle graph is also obtained.

متعددة حدود وينر للجداء التنسري

الملخص

ليكن $G1$ و $G2$ بيانين متصلين وليس بينهما رأس مشترك وان كل حافة فيهما تقع في مثلث. تم في هذا البحث تحديد معاملات متعددة حدود وينر للجداء التنسري $G1\otimes G2$ بدلالة معاملات متعد1دتي حدود وينر $W(G1;x)$ و$W(G2;x)$. وايضا تضمن البحث ايجاد متعددة حدود وينر للجداء التنسري لبيان درب مع بيان دارة فردية.

INTRODUCTION

In this paper, we consider finite connected undirected graphs without loops or multiple edges. For undefined terms, see (Chartrand and Lesniak, 1986) or (Buckley and Harary, 1990).

Let G be a connected non-trivial graph with p vertices and q edges. By the *distance* $d(u,v)$ between the two distinct vertices u and v of G, we mean the length of a shortest path connecting u and v. The diameter, δ, of G is the maximal distance between two of its vertices, that is

$$\delta = \max_{u,v\in V(G)} d(u,v).$$

Let $d(G,k)$ be the number of pairs of vertices in G that are distance k apart, $k = 0, 1, \ldots, \delta$.

It is clear that

$$d(G,0) = p, \quad d(G,1) = q, \quad \text{and} \quad \sum_{k=0}^{\delta} d(G,k) = \binom{p+1}{2}.$$

The Wiener polynomial of G is defined as

$$W(G;x) = \sum_{k=0}^{\delta} d(G,k)x^k, \qquad \ldots\ldots \qquad (1.1).$$

(Hosoya, 1988). The name chosen for $W(G;x)$ honours the physical chemist Harold Wiener who studied distances in graphs and established some of their chemical applications (Wiener, 1947). In the mathematical chemistry literatures, the sum of all distances in a graph G, $\sum_{k=1}^{\delta} kd(G,k)$, is referred to as the Wiener index (or number) and is traditionally denoted by $W(G)$.

In 1993, Gutman established some properties of the Wiener polynomial, and obtained formulas for the Wiener polynomials of the compound graphs $G1\bullet G2$ and $G1{:}G2$, where $G1$ and $G2$ are vertex-disjoint connected graphs. Recently, in 1996, Sagan, Yeh and Zhang described the Wiener polynomials of compound graphs obtained by well-known graph operations such as the join, cartesian product, composition, disjunction, and symmetric difference.

The tensor product of the vertex-disjoint connected graphs $G1=(V1,E1)$ and $G2=(V2,E2)$ is the graph $G1\otimes G2$ defined as

$V(G1\otimes G2) = V1\times V2$,

$E(G1\otimes G2) = \{(u1,v1)\,(u2,v2) \mid u1u2\in E1 \text{ and } v1v2\in E2\}$.

It seems that it is difficult to find $W(G1\otimes G2;x)$ in terms of $W(G1;x)$ and $W(G2;x)$.

In this paper, we obtain a necessary and sufficient condition for $G1\otimes G2$ to be connected. Then we study the Wiener polynomial of $G1\otimes G2$, and find a formula for $W(G1\otimes G2;x)$ when each edge of $G1$ and $G2$ is a triangle edge.

Finally, we obtain the Wiener polynomial of the tensor product of a path graph and an odd cycle graph.

THE CONNECTIVITY OF G1⊗G2

From the definition of tensor product one can easily see that $G1\otimes G2$ may not be connected. For example, $P3\otimes C4$ is disconnected while $P3\otimes C5$ is connected. Therefore, we need to find a necessary and sufficient condition on $G1$ and $G2$ such that $G1\otimes G2$ is connected.

Proposition 1 :

If neither $G1$ nor $G2$ contains an odd cycle, then $G1\otimes G2$ is disconnected.

Proof:

Let $u1u2 \in E1$ and $v\in V2$. We show by contradiction that there is no path joining the two vertices $(u1,v)$, $(u2,v)$, in $G1\otimes G2$. If

P: (u_1,v), (x_1,y_1), (x_2,y_2),…,(x_{r-1},y_{r-1}), (u_2,v) is a (u_1,v) – (u_2,v) path in $G_1 \otimes G_2$, then

Q: u_1, x_1, x_2, …,x_{r-1}, u_2 is a u_1– u_2 walk in G_1 of odd length *r*, for otherwise *Q* with the edge u_1u_2 forms an odd closed walk, which implies the existence of an odd cycle in G_1. Thus

R: v, y_1, y_2,…,y_{r-1}, v

is an odd closed v–v walk in G_2, this means that G_2 contains an odd cycle, a contradiction.

Proposition 2:

If either G_1 or G_2 contains an odd cycle, then $G_1 \otimes G_2$ is connected.

Proof:

We may assume that G_1 contains an odd cycle *C*.

Let (u_1,v_1) and (u_2,v_2) be any two distinct vertices in $G_1 \otimes G_2$. We consider three cases:

(a) If $u_1 \neq u_2$ and $v_1 \neq v_2$, then let P_1 be a u_1–u_2 path in G_1 and P_2 be a v_1–v_2 path in G_2 such that the difference between their lengths is even. Without loss of generality, let $l(P_1) = t$, $l(P_2) = s$, $t \geq s$, and

P_1: $u_1 = x_0, x_1, …, x_t = u_2$,

P_2: $v_1 = y_0, y_2, …, y_s = v_2$.

Then *P*: (x_0,y_0), (x_1,y_1), …, (x_s,y_s), (x_{s+1},y_{s-1}), (x_{s+2},y_s), …, (x_t,y_s), is a (u_1,v_1) – (u_2,v_2) path in $G_1 \otimes G_2$.

If the difference between the length of every u_1–u_2 path in G_1 and every v_1–v_2 path in G_2 is odd, then using the odd cycle *C* we can find a u_1–u_2 walk W_1 in G_1 such $|l(W_1) - l(P_2)|$ is even. Therefore, there is a (u_1,v_1) – (u_2,v_2) walk in $G_1 \otimes G_2$; and so there is a (u_1,v_1) – (u_2,v_2) path in $G_1 \otimes G_2$ (Chartrand and Lesniak, 1986).

(b) If $u_1 \neq u_2$ and $v_1 = v_2$, let *y* be any vertex adjacent with v_1, then, replacing P_2 by the walk

W_2: v_1,, y, v_2,

We can show, as in Case (a), that there is a (u_1,v_1) – (u_2,v_2) path in $G_1 \otimes G_2$.

(c) If $u_1 = u_2$ and $v_1 \neq v_2$, let *x* be any vertex adjacent with u_1, then, replacing P_1 by the walk

W_1: u_1, x, u_1,

We can find, as in Case (a), a (u_1,v_1) – (u_1,v_2) path in $G_1 \otimes G_2$.

Hence, in all cases and for all pairs of vertices (u_1,v_1) and (u_2,v_2) in $G_1 \otimes G_2$, there is a (u_1,v_1)–(u_2,v_2) path. Therefore, $G_1 \otimes G_2$ is connected.

From Propositions 1 and 2, we obtain the following important theorem.

Theorem 3:

Let G_1 and G_2 be disjoint nontrivial connected graphs. Then, $G_1 \otimes G_2$ is connected if and only if either G_1 or G_2 contains an odd cycle.

THE WIENER POLYNOMIAL OF THE TENSOR PRODUCT

In this section, the two graphs G_1 and G_2 are assumed to be nontrivial, connected, disjoint, and either G_1 or G_2 contains an odd cycle.

Lemma 1:

For each pair $(u1,v1)$, $(u2,v2)$ of vertices in $G1 \otimes G2$, we have
$d_{G_1 \otimes G_2}((u_1,v_1),(u_2,v_2)) \geq \max\{d_{G_1}(u_1,u_2), d_{G_2}(v_1,v_2)\}$.

Proof:

Suppose that $d_{G_1 \otimes G_2}((u_1,v_1),(u_2,v_2)) = t$,
and let Q: $(x0,y0)$, $(x1,y1)$, $(x2,y2)$, …,(xt,yt),
be a shortest $(u1,v1)$–$(u2,v2)$ path in $G1 \otimes G2$, in which

$u1 = x0$, $v1 = y0$, $u2 = xt$, $v2 = yt$.

Then, the $x0$–xt walk $x0$, $x1$,…, xt contains a $u1$–$u2$ path of length $\leq t$ in $G1$, and the $y0$–yt walk $y0$, $y1$,…, yt contains a $v1$–$v2$ path of length $\leq t$ in $G2$. Thus $d_{G_1}(u_1,u_2) \leq t$, and $d_{G_2}(v_1,v_2) \leq t$.

This completes the proof.

Definition 2:

An edge e in a graph G is called a triangle edge if there is a cycle of length 3 in G containing e.

Lemma 3:

Let each edge of $G2$ be a triangle edge. If $d_{G_1}(u_1,u_2) \geq 2$, and
$d_{G_1}(u_1,u_2) \geq d_{G_2}(v_1,v_2)$, then

$$d_{G_1 \otimes G_2}((u_1,v_1),(u_2,v_2)) \leq d_{G_1}(u_1,u_2).$$

Proof:

It is obvious that $u1 \neq u2$. Let

$$d_{G_1}(u_1,u_2) = t, \qquad d_{G_2}(v_1,v_2) = s,$$

and let

$Q1$: $x0$, $x1$, $x2$, …, xt, where $x0=u1$, $xt=u2$,
$Q2$: $y0$, $y1$, $y2$, …, ys, where $y0=v1$, $ys=v2$,

be a shortest $u1$–$u2$ path in $G1$ and a shortest $v1$–$v2$ path in $G2$, respectively. We consider two cases:

Case I:

$t - s$ is even.

If $s > 0$, then

$(x0,y0)$, $(x1,y1)$, …, (xs,ys), $(xs+1,ys-1)$, $(xs+2,ys)$, …, (xt,ys)

is a $(u1,v1)$ – $(u2,v2)$ path in $G1 \otimes G2$ of length t.

If $s = 0$, that is $y0 = v1= v2$, then let y be any vertex adjacent to $v1$. Then

$(x0,y0)$, $(x1,y)$, $(x2,y0)$, $(x3,y)$, …, $(xt-1,y)$, $(xt,y0)$

is a $(u1,v1)$ – $(u2,v1)$ path in $G1 \otimes G2$ of length t.

Case II:

$t - s$ is odd.

If $s > 0$, let $y0y1z$ be the triangle containing the edge $y0y1$ in $G2$. Then

$(x0,y0)$,$(x1,z)$,$(x2,y1)$,…,$(xs,ys-1)$,$(xs+1,ys)$,$(xs+2,ys-1)$, $(xs+3,ys)$,…,(xt,ys).

is a $(u1,v1) – (u2,v2)$ path in $G1\otimes G2$ of length t.
If $s = 0$, then t is odd and $t \geq 3$. In this case, let $v1yz$ be the triangle containing vertex $v1$ ($=y0$).
Then

$(x0,y0), (x1,y), (x2, z), (x3,y0), (x4,y), (x5,y0),\ldots, (xt,y0)$

is a $(u1,v1) – (u2,v1)$ path of length t in $G1\otimes G2$.
Therefore, in all cases, $d_{G_1\otimes G_2}((u_1,v_1),(u_2,v_2)) \leq t$.

Hence, the proof is completed.

Similarly, we can prove that if each edge of $G1$is a triangle edge, and

$$d_{G_2}(v_1,v_2) \geq 2,$$

and $d_{G_1}(u_1,u_2) \leq d_{G_2}(v_1,v_2)$,

then $d_{G_1\otimes G_2}((u_1,v_1),(u_2,v_2)) \leq d_{G_2}(v_1,v_2)$.

Moreover, if each edge of $G1$ and $G2$ is a triangle edge, then

$$d_{G_1\otimes G_2}((u_1,v),(u_2,v)) = 2 \quad if \quad u_1u_2 \in E_1,$$

and $d_{G_1\otimes G_2}((u,v_1),(u,v_2)) = 2 \quad if \quad v_1v_2 \in E_2$.

Hence, by Lemmas 1 and 3, we have the following theorem.

Theorem 4:

Let $G1$ and $G2$ be nontrivial disjoint connected graphs such that each edge of $G1$ and $G2$ is a triangle edge, then

$$d_{G_1\otimes G_2}((u_1,v_1),(u_2,v_2))=\begin{cases} 2\ , & when (u_1=u_2 \ and\ v_1v_2\in E_2)\ or\ (v_1=v_2\ and\ u_1u_2\in E_1) \\ \max\{d_{G_1}(u_1,u_2),d_{G_2}(v_1,v_2)\}, & otherwise \end{cases}$$

Corollary 5:

If each edge of $G1$ and $G2$ is a triangle edge, then

$$diam(G_1\otimes G_2)=\begin{cases} \max\ \{diamG_1,\ diamG_2\}=\delta, & when\ \delta\geq 2 \\ 2, when\ G_1\ and\ G_2\ are\ complete\ graphs. \end{cases}$$

The proof follows from Theorem 4.

Theorem 6:

Let $G1$ and $G2$ be nontrivial disjoint graphs which are not both complete graphs. If each edge of $G1$ and $G2$ is a triangle edge, and

$$W(G_1;x)=\sum_{i=0}^{\delta_1} a_i x^i, \quad W(G_2;x)=\sum_{i=0}^{\delta_2} b_i x^i,$$

in which $\delta 1 = diamG1$ and $\delta 2 = diamG2$,

then

$$W(G_1\otimes G_2;x)=\sum_{i=0}^{\delta} c_i x^i,$$

where

$\delta = max\ \{\delta 1,\delta 2\}$

$c_0=a_0b_0$,
$c_1=2a_1b_1$,
$c_2=a_0(b_1+b_2) + a_1(b_0+2b_2) + a_2(b_0+2b_1+2b_2)$,
and for $k \geq 3$,
$$c_k=a_k(b_0+2b_1+\ldots+2b_{k-1}+b_k)+b_k(a_0+2a_1+\ldots+2a_{k-1}+a_k).$$

Proof:

Let (u_1,v_1), (u_2,v_2) be two vertices of $G_1\otimes G_2$ that are distance k (≥ 3) apart. Then by Theorem 3.4 either

(1) $d_{G_1}(u_1,u_2)=k$ and $0\leq d_{G_2}(v_1,v_2)\leq k$,
which gives $a_k(b_0 + b_1 +\ldots+b_k)$ such pairs;

or

(2) $d_{G_2}(v_1,v_2)=k$ and $0\leq d_{G_1}(u_1,u_2)\leq k$,
which gives $b_k(a_0 + a_1 + \ldots +a_k)$ such pairs.

Moreover, if $u_1 \neq u_2$ and $v_1 \neq v_2$, then
$$d_{G_1\otimes G_2}((u_1,v_1),(u_2,v_2)) = d_{G_1\otimes G_2}((u_1,v_2),(u_2,v_1)).$$

Thus, the total number of pairs of vertices that are distance $k(\geq 3)$ apart is
$c_k = a_k(b_0 + 2 b_1 + 2b_2 +\ldots+ 2b_{k-1} + b_k) + b_k(a_0 + 2 a_1 + 2a_2 +\ldots+ 2a_{k-1} + a_k)$.

For $k = 2$, we have
$$c_2 = a_2(b_0 + 2 b_1 + b_2) + b_2(a_0 + 2a_1 + a_2) + a_0b_1 + a_1b_0$$
$$= a_0(b_1 + b_2) + a_1(b_0 + 2b_2) +a_2(b_0 + 2b_1 +2b_2),$$
which completes the proof.

Corollary 7:

If K_{p_1} and K_{p_2} are disjoint complete graphs with $p_1,p_2 \geq 3$, then
$$W(K_{p_1}\otimes K_{p_2};x) = \frac{1}{2}p_1p_2\{2+(p_1-1)(p_2-1)x + (p_1+p_2-2)x^2\}. \blacksquare$$

THE WIENER POLYNOMIAL OF THE TENSOR PRODUCT OF A PATH AND AN ODD CYCLE

Consider the tensor product of the path P_n, $n \geq 2$, and the odd cycle C_{2m+1}, $m \geq 1$. By Theorem 2.3, the graph $G = P_n\otimes C_{2m+1}$ is connected.

Let
$$V(P_n) = \{u_0,u_1,u_2, \ldots,u_{n-1}\},\ V(C_{2m+1}) = \{v_0,v_1,v_2, \ldots,v_{2m}\}.$$
(See Figure 1)

It is clear that
$$d(u_i,u_j) = |j-i|, \text{ and } d(v_i,v_j) = min\ \{ |j-i|, 2m+1 - |i-j| \}$$

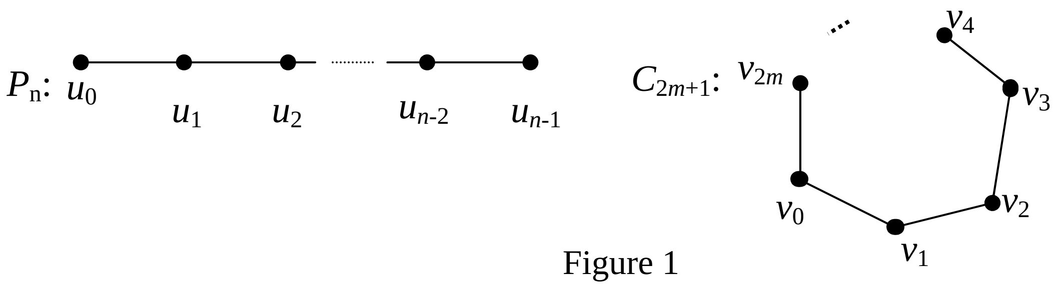

Figure 1

From Lemma 1:

$dG((ui,vs),(uj,vt)) \geq max\ \{d(ui,uj), d(vs,vt)\}$. …… (1)

Thus, if

$R = |d(ui,uj)- d(vs,vt)|$ …… (2)

is even, then as in Case I of the proof of Lemma 3.3,

$dG((ui,vs),(uj,vt)) = max\ \{d(ui,uj), d(vs,vt)\}$. …… (3)

But, if R is odd, then

$|d(ui,uj)- \{2m+1- d(vs,vt)\}|$

is even. Hence,

$dG((ui,vs),(uj,vt)) \leq max\ \{d(ui,uj), 2m+1- d(vs,vt)\}$…..(4)

Now, let $(ui,vs),(x1,y1), …,(xh,yh),(uj,vt)$ be a shortest (ui,vs)- (uj,vt) path in $Pn\otimes C2m+1$, then the walks:

$F1$: $ui, x1, …, xh, uj$ in Pn,

and

$F2$: $vs, y1, …, yh, vt$ in $C2m+1$,

have the same length. Let $Q1$ be the ui-uj path in Pn, and let $Q2$ be the vs–vt path in $C2m+1$ which is contained in $F2$. Then $l(F1)$- $l(Q1)$ is an even integer because Pn is acyclic graph. Also, $l(F2)$ - $l(Q2)$ is an even integer because $F2$ contains no cycles. (If $F2$ contains $C2m+1$ itself, then the inequality (5) holds immediately). Therefore $l(Q1)$-$l(Q2)$ is an even integer. Obviously,

$l(Q1) = d(ui,uj)$.

Because R is odd, then

$l(Q2) \neq d(vs,vt)$, which means that $l(Q2) = (2m+1) - d(vs,vt)$.

Thus

$dG((ui,vs),(uj,vt)) \geq max\ \{d(ui,uj), 2m+1- d(vs,vt)\}$… (5)

Hence, from (4.3), (4.4), and (4.5), we have the following result.

Proposition 1:

Let $G = Pn\otimes C2m+1, n \geq 2, m \geq 1$, then

$$d((u_i,v_s),(u_j,v_t)) = \begin{cases} \max\{d(u_i,u_j), d(v_s,v_t)\}, if\ \ R\ \ is\ \ even; \\ \max\{d(u_i,u_j), 2m+1-d(v_s,v_t)\}, if\ \ R\ \ is\ \ odd. \end{cases}$$ in which

$R = |d(ui,uj)- d(vs,vt)|$. ∎

Corollary 2:

$$diam(P_n \otimes C_{2m+1}) = \max\{n-1, 2m+1\}.$$

Proof:

From Proposition 4.1, for each pair (ui,vs), $(u\,j, vt)$

$dG((ui,vs), (u\,j, vt)) \leq \max\ \{n-1,2m+1\}$, …… (6)

and

$dG((u0,vs), (u\ n-1, vs) = \max\ \{n-1,2m+1\}$,

when n is even. Thus if n is even, then

$diam(Pn\otimes C2m+1) = \max\ \{n-1,2m+1\}$.

If n is odd, then n-1 is even, and

$d_G((u_0,v_s), (u_{n-1}, v_s)) = n-1$, and
$d_G((u_0,v_s), (u_1, v_s)) = 2m+1$
Hence, the proof follows from (4.6). ▬
By Theorem 1.2 (5,6) of (Sagan, Yeh and Zhang, 1996).

$$W(C_{2m+1};x) = (2m+1)\sum_{i=0}^{m} x^i$$

and

$$W(P_n;x) = \sum_{i=0}^{n-1} (n-i)x^i .$$

Let $\delta = \max\{n-1, 2m+1\}$, and

$$W(P_n \otimes C_{2m+1};x) = \sum_{i=0}^{\delta} c_i x^i ,$$

then $c_0 = n(2m+1)$; $c_1 = 2(n-1)(2m+1)$.
To find c_i for $2 \le i \le \delta$, we consider several cases.

Proposition 3:

$$c_\delta = (2m+1)\begin{cases} (2m+1)^2, & when \ \ n-1 \ge 2m+1; \\ \left(\frac{n}{2}\right)^2, & when \ \ n-1 < 2m+1 \ \ and \ \ n \ \ is \ \ even; \\ \frac{1}{4}(n^2-1), & when \ \ n-1 < 2m+1 \ \ and \ \ n \ \ is \ \ odd. \end{cases}$$

Proof:

If $n-1 \ge 2m+1$, then $\delta = n-1$, and a pair in $P_n \otimes C_{2m+1}$ is of distance $n-1$ apart if and only if it is of the form (u_0,v_s), (u_{n-1},v_t) for each pair $u_s,v_t \in V(C_{2m+1})$ including the cases when $v_s=v_t$. Thus, the total number of such pairs is $(2m+1)^2$.

If $(n-1) < (2m+1)$, then a pair in $P_n \otimes C_{2m+1}$ is of distance $(2m+1)$ apart if and only if it is of the form (u_i,v_s), (u_j,v_s) for which $d(u_i,u_j)$ is odd and for all $v_s \in V(C_{2m+1})$. When n is even the total number of such pairs is

$$(2m+1)\,[\,(n-1) + (n-3) + \dots + 1] = (2m+1)\left(\frac{n}{2}\right)^2 .$$

And, when n is odd, the total number of pairs is

$$(2m+1)\,[\,(n-1) + (n-3) + \dots + 2] = (2m+1)\left(\frac{n^2-1}{4}\right).$$

Hence, the proof is completed. ▬

Theorem 4:

The coefficients, c_k $2 \le k < \delta$, of the Wiener polynomial $W(P_n \otimes C_{2m+1};x)$ are given by:

(a) if k is even and $k \le n-1$, then $c_k = (2m+1)(2nk - \frac{3}{2}k^2)$.

(b) If k is even and $k > n-1$, then

$$c_k = \frac{1}{2}(2m+1)\begin{cases} n^2, & when \quad n \quad is \quad even, \\ n^2+1, & when \quad n \quad is \quad odd. \end{cases}$$

(c) If k is odd and $k \le n-1$, then $c_k = (2m+1)[2nk - \frac{1}{2}(3k^2+1)]$.

(d) If k is odd and $k > n-1$, then

$$c_k = \frac{1}{2}(2m+1)\begin{cases} n^2, & when \quad n \quad is \quad even, \\ n^2-1, & when \quad n \quad is \quad odd. \end{cases}$$

Proof:

Case (a):

We have two possibilities:

(i) $k \le m$, and (ii) $k > m$.

(i) If $k \le m$, then by Proposition 4.1, a pair (u_i,v_s), (u_j,v_t) of $P_n \otimes C_{2m+1}$ is of distance k apart in the following two subcases:

(1) $d(u_i,u_j) = k$, and $d(v_s,v_t)$ is even with $d(v_s,v_t) \le k$.

(2) $d(v_s,v_t) = k$, and $d(u_i,u_j)$ is even with $d(u_i,u_j) < k$.

Thus, the total number of such pairs in the two subcases is

$$(n-k)(2m+1)(k+1) + (2m+1)[n+2(n-1)+2(n-4)+\ldots+2(n-(k-2))]$$

$$= (2m+1)(2nk - \frac{3}{2}k^2).$$

(ii) If $k > m$, then by Proposition 4.1, a pair (u_i,v_s), (u_j,v_t) of $P_n \otimes C_{2m+1}$ is of distance k apart in the following three subcases:

(1) $d(u_i,u_j) = k$, $d(v_s,v_t) \le m$ and $d(v_s,v_t)$ is even.

(2) $d(u_i,u_j) = k$, $2m+1-k < d(v_s,v_t) \le m$ and $d(v_s,v_t)$ is odd.

(3) $d(u_i,u_j) < k$, $d(v_s,v_t) = 2m+1-k$ and $d(u_i,u_j)$ is even.

Hence, the total number of such pairs in the three subcases is

$$(n-k)(2m+1)\begin{cases} (m+1), & when \quad m \quad is \quad even, \\ m, & when \quad m \quad is \quad odd. \end{cases}$$

$$+\ 2(n-k)(2m+1)\left\lceil \frac{k-m}{2} \right\rceil$$

$$+ (2m+1)[n+2(n-2)+2(n-4)+\ldots+2(n-(k-2))]$$

$$= (n-k)(2m+1)(k+1)+(2m+1)\left[n + 2\left(\frac{2n-k}{2}\right)\left(\frac{k-2}{2}\right) \right]$$

$= (2m+1)(2nk - \frac{3}{2}k2)$.

This completes the proof of Case (a).

Case (b):

As in Case (a), we have two possibilities:

(i) If $k \le m$, then by Proposition 4.1, a pair (ui,vs), (uj,vt) is of distance k apart if and only if $d(ui,uj) \le n$-1, $d(ui,uj)$ even and $d(vs,vt) = k$.

The number of such pairs is given by

$$(2m+1)\begin{cases}[n+2(n-2)+...+2(n-(n-2)), & when \quad n \quad is \quad even,\\ [n+2(n-2)+...+2(n-(n-1)), & when \quad n \quad is \quad odd.\end{cases}$$

$$=\frac{1}{2}(2m+1)\begin{cases}n^2, & when \quad n \quad is \quad even,\\ (n^2+1), & when \quad n \quad is \quad odd.\end{cases}$$

(ii) If $k > m$, then a pair (ui,vs), (uj,vt) is of distance k apart if and only if $2m+1- d(vs,vt) = k$, $d(ui,uj)$ is even, and $d(ui,uj) \le n$-1.

Thus, the number of such pairs is exactly that given in (i) of this case.

This completes the proof of Case (b).

Case (c):

We have two possiblities:

(i) If $k \le m$, then a pair (ui,vs), (uj,vt) of $Pn\otimes C2m+1$ is of distance k apart in the following two subcases:

(1) $d(ui,uj) = k$, $d(vs,vt) \le k$ and $d(vs,vt)$ is odd.

(2) $d(vs,vt) = k$, $d(ui,uj) < k$ and $d(ui,uj)$ is odd.

Therefore, the total number of such pairs is given by:

$$(n-k)(2m+1)(k+1) + (2m+1)(2)[(n-1)+(n-3)+\ldots+(n-(k-2))]$$

$$= (2m+1)[(2nk - \frac{1}{2}(3k2+1)].$$

(ii) If $k > m$, then a pair (ui,vs), (uj,vt) of $Pn\otimes C2m+1$ is of distance k apart in the following subcases:

(1) $d(ui,uj) = k$, $d(vs,vt) \le m$ and $d(vs,vt)$ is odd.

(2) $d(ui,uj) = k$, $2m+1-k < d(vs,vt) \le m$ and $d(vs,vt)$ is even.

(3) $d(ui,uj) < k$, $d(vs,vt)= 2m+1-k$ and $d(ui,uj)$ is odd.

Hence, the total number of such pairs is given by

$$2(n-k)(2m+1)\begin{cases}\frac{m}{2}, & when \quad m \quad is \quad even,\\ (\frac{m+1}{2}), & when \quad m \quad is \quad odd.\end{cases}$$

$$+ 2(n-k)(2m+1)\left\lceil\frac{k-m}{2}\right\rceil$$

$$+ (2m+1)[2(n-1)+2(n-3)+\ldots+2(n-(k-2))]$$

$= (n\text{-}k)(2m+1)(k+1)+2(2m+1)\left[\left(\frac{2n-k+1}{2}\right)\left(\frac{k-1}{2}\right)\right]$

$= (2m+1)[2nk - \frac{1}{2}(3k2+1)]$.

This completes the proof of Case (c).

Case (d):

Here, also we consider two possibilities:

(i) If $k \le m$, then a pair (ui,vs), (uj,vt) is of distance k apart if and only if $d(ui,uj) \le n$-1, $d(vs,vt) = k$, and $d(ui,uj)$ is odd.

Thus, the number of such pairs is given by

$$(2m+1)(2)\begin{cases}[(n-1)+(n-3)+...+(n-(n-1))], when\ \ n\ \ is\ \ even,\\ [(n-1)+(n-3)+...+(n-(n-2))], when\ \ n\ \ is\ \ odd.\end{cases}$$

$$=\frac{1}{2}(2m+1)\begin{cases}n^2, & when\ \ n\ \ is\ \ even,\\ (n^2-1), & when\ \ n\ \ is\ \ odd.\end{cases}$$

(ii) If $k > m$, then a pair (ui,vs), (uj,vt) is of distance k apart if and only if $d(vs,vt) = 2m+1\text{-}\ k$, $d(ui,uj) \le n$-1, and $d(ui,uj)$ is odd.

Thus, the number of such pairs is given by:

$$(2m+1)(2)\begin{cases}[(n-1)+(n-3)+...+(n-(n-1))], when\ \ n\ \ is\ \ even,\\ [(n-1)+(n-3)+...+(n-(n-2))], when\ \ n\ \ is\ \ odd.\end{cases}$$

$$=\frac{1}{2}(2m+1)\begin{cases}n^2, & when\ \ n\ \ is\ \ even,\\ (n^2-1), & when\ \ n\ \ is\ \ odd.\end{cases}$$

This completes the proof of Case (d).

Hence, the proof of theorem.

REFERENCES

Buckley, F. and Harary, F., 1990. Distance in Graphs, Addison-Wesley, Redwood.

Chartrand, G. and Lesniak, L., 1986. Graphs and Digraphs, 2nd ed., Wadsworth and Brooks/ Cole, California.

Gutman, I., 1993. Some properties of the Wiener polynomial, Graph Theory Notes of New York, XXV, The New York Academy of Sciences, pp.13-18.

Hosoya, H., 1988. On some counting polynomials in chemistry, Discrete Applied Math., Vol.19, pp.239-257.

Sagan, B.E., Yeh, Y.N. and Zhang, P., 1996. The Wiener polynomial of a graph, international J. Quantum Chem., Vol. 60, No.5, pp.959-969.

Wiener, H., 1947. Structural determination of paraffin boiling points, J. Amer. Chem. Soc., Vol. 69, pp.17-20.